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COMPLEX LAPLACIANS ON ALMOST-HERMITIAN MANIFOLDS

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Introduction

In [2] Hsiung (i) defined a new complex Laplacian $\square_2$ for an almost-Hermitian structure, which is different from the one, denoted by $\square_1$, given by Kodaira and Spencer [3], (ii) verified for $\square_2$ the well-known conjecture that if $\square_2 = \Delta/2$ for all 0- and 1-forms, where Δ is the real Laplacian, then the structure is Kählerian, (iii) studied the conditions for $\square_2$ to be real for all 0- and 1-forms. Very recently, Ogawa [5] continued Hsiung's work to show that if either $\square_2$ or $\square_1$ is real for all 0- and 1-forms, then the structure is Kählerian.

The purpose of this paper is to introduce three more complex Laplacians $\square_3$, $\square_4$, $\square_5$ for an almost-Hermitian structure and to study the conditions for these Laplacians to be real, together with some relationships among all $\square$'s. We shall continue to use Hsiung's method [2] which is somewhat different from Ogawa's, and also for completeness we shall reprove Ogawa's result here.

§ 1 contains fundamental notation and real operators on a Riemannian manifold. In § 2 we define various almost-Hermitian structures first and then some complex operators for an almost Hermitian structure leading to the complex Laplacians $\square_i, i = 1, \cdots, 5$. Some conditions for the tensor of an almost-Hermitian structure to be Kählerian are also given for use in the proofs of our main theorems. § 3 is devoted to the computation of $\square_i\xi$ and $\square_i\eta, i = 1, \cdots,$ 5, for any 0-form ξ and 1-form η on an almost-Hermitian manifold. In § 4 we show that for an almost-Hermitian structure if the complex Laplacian $\square_i$, $i = 1, 2$ or 4 is real with respect to all 0- and 1-forms, then the structure is Kählerian. In § 5 we obtain the following relationships among the $\square$'s: If for an almost-Hermitian structure the relation Im $\square_1 =$ Im $\square_i$ ($i = 2$ or 4) or Im $\square_2 =$ Im $\square_j$ ($j = 4$ or 5) holds for all 0- and 1-forms, where Im denotes the imaginary part, then the structure is Kählerian.

Throughout this paper, the dimension of a manifold M^n is understood to be $n \geq 2$, and all forms and structures are of class at least C^2.

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1. Notation and real operators

Let M^n be a Riemannian manifold of dimension $n(\geq 2)$, $\| g_{ij} \|$ with $g_{ij} = g_{ji}$ be the matrix of the positive definite metric of M^n, and $\| g^{ij} \|$ be the inverse matrix of $\| g_{ij} \|$. Throughout this paper all Latin indices take the values $1, \cdots, n$ unless stated otherwise. We shall follow the usual tensor convention that indices can be raised and lowered by using g^{ij} and g_{ij} respectively, and also that when a Latin letter appears in any term as a subscript and superscript, it is understood that this letter is summed over its range. Moreover, if we multiply, for example, the components a_{ij} of a tensor of type $(0,2)$ by the components b^{jk} of a tensor of type $(2,0)$, it will always be understood that j is to be summed.

Let $\mathcal{N}$ be the set $\{1, \cdots, n\}$ of positive integers less than or equal to n, and let $I(p)$ denote an ordered subset $\{i_1, \cdots, i_p\}$ of the set $\mathcal{N}$ for $p \leq n$. If the elements $i_1, \cdots, i_p$ are in the natural order, that is, if $i_1 < \cdots < i_p$, then the ordered set $I(p)$ is denoted by $I_0(p)$. Furthermore, denote the nondecreasingly ordered p-tuple having the same elements as $I(p)$ by $\langle I(p) \rangle$, and let $I(p; \hat{s} \mid j)$ be the ordered set $I(p)$ with the s-th element i_s replaced by another element j of $\mathcal{N}$, which may or may not belong to $I(p)$. We shall use these notations for indices throughout this paper. When more than one set of indices is needed at one time, we may use other capital letters such as $J, K, L, \cdots$ in addition to I.

At first we define

$$(1.1) \qquad \varepsilon^{J(p)}_{K(p)} = \begin{cases} 0, & \text{if } \langle J(p) \rangle \neq \langle K(p) \rangle, \\ 0, & \text{if } J(p) \text{ or } K(p) \text{ contains repeated integers,} \\ +1 \text{ or } -1, & \text{if the permutation taking } J(p) \text{ into } K(p) \text{ is} \\ & \quad \text{even or odd.} \end{cases}$$

By counting the number of terms it is easy to verify that

$$(1.2) \qquad \varepsilon^{I(p)J(n-p)}_{1\cdots n} \varepsilon^{1\cdots n}_{I(p)K(n-p)} = p!\, \varepsilon^{J(n-p)}_{K(n-p)} ,$$

$$(1.3) \qquad \varepsilon^{I(p)J(q)}_{K(p+q)} \varepsilon^{L(p)}_{I(p)} = p!\, \varepsilon^{L(p)J(q)}_{K(p+q)} .$$

On the manifold M^n, let ∇ denote the covariant derivation with respect to the affine connection Γ, with components Γ^i_{jk} in local coordinates $x^1, \cdots, x^n$, of the Riemannian metric g, and let ϕ be a differential form of degree p given by

$$(1.4) \qquad \phi = \frac{1}{p!} \phi_{I(p)} dx^{I(p)} = \phi_{I_0(p)} dx^{I_0(p)} ,$$

where $\phi_{I(p)}$ is a skew-symmetric tensor of type $(0, p)$, and we have placed

$$(1.5) \qquad dx^{I(p)} = dx^{i_1} \wedge \cdots \wedge dx^{i_p} .$$

Then we have

(1.6) $$d\phi = (d\phi)_{I_0(p+1)} dx^{I_0(p+1)} ,$$

where

(1.7) $$(d\phi)_{I(p+1)} = \frac{1}{p!} \varepsilon^{kJ(p)}_{I(p+1)} \nabla_k \phi_{J(p)} .$$

Denote

(1.8) $$e_{I(n)} = \varepsilon^{1\cdots n}_{I(n)} (\det (g_{ij}))^{1/2} .$$

Then by using orthonormal local coordinates $x^1, \cdots, x^n$ and relation (1.2) we can easily obtain

(1.9) $$e_{I(p)K(n-p)} e^{I(p)J(n-p)} = p!\, \varepsilon^{J(n-p)}_{K(n-p)} .$$

The dual operator $*$ is defined by (see, for instance, [6])

(1.10) $$* \phi = (* \phi)_{I_0(n-p)} dx^{I_0(n-p)} ,$$

where

(1.11) $$(* \phi)_{I(n-p)} = \frac{1}{p!} e_{J(p)I(n-p)} \phi^{J(p)} .$$

From (1.10), (1.11) it follows that for the scalar 1

(1.12) $$* 1 = (\det (g_{ij}))^{1/2} dx^1 \wedge \cdots \wedge dx^n ,$$

which is just the element of area of the manifold M^n. By using orthonormal local coordinates $x^1, \cdots, x^n$ we can easily verify that

(1.13) $$** \phi = (-1)^{p(n-p)} \phi .$$

Denote the inverse operator of $*$ by $*^{-1}$. Then from (1.13) it is seen that on forms of degree p

(1.14) $$*^{-1} = (-1)^{p(n-p)} * .$$

The codifferential operator δ is defined by

(1.15) $$\delta\phi = (-1)^{p+n+1} *^{-1} d * \phi .$$

Making use of (1.6), (1.7), (1.10), (1.11) we obtain immediately

(1.16) $$\delta\phi = (\delta\phi)_{I_0(p-1)} dx^{I_0(p-1)} ,$$

where

$$(\delta\phi)_{I(p-1)} = -\nabla_j \phi^j{}_{I(p-1)} \,. \tag{1.17}$$

For a form ϕ of degree p defined by (1.4) we can obtain

$$\begin{aligned} p!\,(\Delta\phi)_{I(p)} = &-\nabla^j\nabla_j\phi_{I(p)} + \sum_{s=1}^{p} \phi_{I(p;\hat{s}|a)} R^a{}_{i_s} \\ &+ \sum_{s<t}^{1,\cdots,p} \phi_{I(p;\hat{s}|a,\hat{t}|b)} R^{ab}{}_{i_s i_t} \,, \end{aligned} \tag{1.18}$$

where Δ is the Laplace-Beltrami operator defined by

$$\Delta = \delta d + d\delta \,, \tag{1.19}$$

and

$$\nabla^j = g^{jk}\nabla_k \,, \tag{1.20}$$

$$R^i{}_{jkl} = \partial\Gamma^i_{jk}/\partial x^l - \partial\Gamma^i_{jl}/\partial x^k + \Gamma^h_{jk}\Gamma^i_{hl} - \Gamma^h_{jl}\Gamma^i_{hk} \,, \tag{1.21}$$

$$R_{jk} = R^s{}_{jks} \,. \tag{1.22}$$

2. Complex structures and operators

On a Riemannian manifold M^n with metric tensor g_{ij}, if there exists a tensor $F_i{}^j$ of type (1,1) satisfying

$$F_i{}^j F_j{}^k = -\varepsilon_i^k \,, \tag{2.1}$$

then $F_i{}^j$ is said to define an almost-complex structure on the manifold M^n, and the manifold M^n is called an almost-complex manifold. From (2.1) it follows that the almost-complex structure $F_i{}^j$ induces an automorphism J of the tangent space of the manifold M^n at each point with $J^2 = -I$, I being the identity operator, such that, for any tangent vector v^k,

$$J:\, v^k \to F_i{}^k v^i \,. \tag{2.2}$$

If an almost-complex structure $F_i{}^j$ further satisfies

$$g_{ij} F_h{}^i F_k{}^j = g_{hk} \,, \tag{2.3}$$

then $F_i{}^j$ is said to define an almost-Hermitian structure on the manifold M^n, and the manifold M^n is called an almost-Hermitian manifold. From (2.1), (2.3) it follows that the tensor F_{ij} of type (0,2) defined by

$$F_{ij} = g_{jk} F_i{}^k \tag{2.4}$$

is skew-symmetric. Thus on an almost-Hermitian manifold we have the associated differential form

$$\omega = F_{ij} dx^i \wedge dx^j \; . \tag{2.5}$$

By using the multiplication of matrices, from (2.1) we readily see that a necessary condition for the existence of an almost-complex structure on a Riemannian manifold M^n is that the dimension n of the manifold M^n be even. It should also be remarked that an almost-complex manifold is always orientable, and the orientation depends only on the tensor $F_i{}^j$.

An almost-Hermitian structure $F_i{}^j$ defined on a manifold M^n is called an almost-Kählerian structure and the manifold M^n an almost-Kählerian manifold, if the associated form ω is closed, that is,

$$d\omega = 0 \; . \tag{2.6}$$

From (2.5), (2.6) it follows that an almost-Kählerian structure $F_i{}^j$ satisfies

$$F_{hij} \equiv \nabla_h F_{ij} + \nabla_i F_{jh} + \nabla_j F_{hi} = 0 \; . \tag{2.7}$$

The tensor F_{hij} is obviously skew-symmetric in all indices.

An almost-Hermitian structure $F_i{}^j$ (respectively manifold) satisfying

$$F_i \equiv -\nabla_j F_i{}^j = 0 \tag{2.8}$$

is called an almost-semi-Kählerian structure (respectively manifold). In particular, the structure $F_i{}^j$ is Kählerian if $\nabla_i F_j{}^k = 0$. In this case, by means of (2.1) it is easily seen that the torsion tensor

$$t_{ij}{}^k = F_j{}^h(\partial F_i{}^k/\partial x^h - \partial F_h{}^k/\partial x^i) - F_i{}^h(\partial F_j{}^k/\partial x^h - \partial F_h{}^k/\partial x^j)$$

vanishes, so that the integrability condition of the almost-complex structure $F_i{}^j$ is satisfied. But in general when $t_{ij}{}^k = 0$, the almost-Hermitian structure $F_i{}^j$ is defined to be Hermitian

Multiplying (2.4) by F^{hi} we obtain

$$F_{ij} F^{hi} = -\varepsilon_j^h \; . \tag{2.9}$$

By taking covariant differentiation of both sides of (2.9), noticing that

$$F^{ij} \nabla_h F_{ij} = 0 \; , \tag{2.10}$$

and making use of (2.7), (2.8) it is easily seen that

$$F_{hij} F^{ij} = 2 F_h{}^i F_i \; . \tag{2.11}$$

Thus an almost-semi-Kählerian structure $F_i{}^j$ satisfies

$$F_{hij} F^{ij} = 0 \; . \tag{2.12}$$

Multiplication of (2.11) by $F_k{}^h$ and use of (2.9) give

$$F_k = -\tfrac{1}{2}F_{hij}F^{ij}F_k{}^h \ . \tag{2.13}$$

From (2.7), (2.8), (2.13) we hence conclude that *an almost-Kählerian structure or manifold is also almost-semi-Kählerian.*

In the proofs of our theorems we shall need the following lemmas.

Lemma 2.1. *An almost-Hermitian structure F satisfying*

$$\nabla_i F_{jk} = \nabla_j F_{ik} \tag{2.14}$$

is Kählerian.

Proof. From the skew-symmetry of F_{ij} we have

$$\nabla_i F_{jk} + \nabla_i F_{kj} = 0 \ . \tag{2.15}$$

Taking the sum of (2.15) and the two similar equations obtained from it by cyclic permutation of the indices i, j, k, and making use of (2.14) we obtain $\nabla_i F_{jk} + \nabla_i F_{kj} + \nabla_j F_{ki} = 0$, which together with (2.15) implies immediately $\nabla_j F_{ki} = 0$.

Lemma 2.2. *An almost-Hermitian structure F satisfying*

$$F^{ij}\nabla^k \nabla_k F_{ij} = 0 \tag{2.16}$$

is Kählerian.

Proof. From (2.9) we have

$$0 = \nabla^k \nabla_k (F_{ij}F^{ij}) = 2(F^{ij}\nabla^k \nabla_k F_{ij} + \nabla_k F_{ij} \nabla^k F^{ij}) \ ,$$

which together with (2.16) gives $\nabla_k F_{ij} \nabla^k F^{ij} = 0$ and therefore $\nabla_k F_{ij} = 0$.

Lemma 2.3 (*S. Kotô* [4]). *An almost-Hermitian structure F satisfying*

$$\nabla_i F_j{}^k + \nabla_j F_i{}^k = 0 \ , \tag{2.17}$$

$$R_{hi} = -\tfrac{1}{2}R_{hjkl}F^{kl}F_i{}^j \tag{2.18}$$

is Kählerian.

Proof. (2.17) can be written as

$$\nabla_i F_{jk} = \nabla_k F_{ij} \ . \tag{2.19}$$

Multiplying (2.19) by F^{ij}, using (2.10) and taking the covariant derivative ∇_l of the resulting equation, we obtain, in consequence of (2.19),

$$F^{ij}\nabla_l \nabla_i F_{jk} + \nabla_k F_{ij} \nabla_l F^{ij} = 0 \ . \tag{2.20}$$

On the other hand, using (2.19) and the relation $-F^{ij}\nabla_j\nabla_i F_{kl} = F^{ij}\nabla_i\nabla_j F_{kl}$, from the Ricci identity it follows respectively that

$$\nabla_l\nabla_i F_{jk} = \nabla_i\nabla_j F_{kl} + R_{ajil}F_k{}^a - R_{akil}F_j{}^a \,, \tag{2.21}$$

$$F^{ij}\nabla_i\nabla_j F_{kl} = -\tfrac{1}{2}F^{ij}(R^a{}_{kji}F_{al} + R^a{}_{lji}F_{ka}) \,. \tag{2.22}$$

Similarly, the Bianchi identity leads to

$$\begin{aligned} 2R_{hijk}F^{ij} &= R_{hijk}F^{ij} - R_{hjik}F^{ij} \\ &= (R_{hjik} + R_{hikj})F^{ij} - 2R_{hkij}F^{ij} \,, \end{aligned}$$

and therefore to

$$R_{hijk}F^{ij} = -\tfrac{1}{2}R_{hkij}F^{ij} \,. \tag{2.23}$$

Substituting (2.21) in (2.20) and using (2.22), (2.23), (2.1) we can obtain

$$\nabla_k F_{ij}\nabla_l F^{ij} = R_{kl} + F^{ij}(\tfrac{1}{2}R_{akij}F_l{}^a - R_{alij}F_k{}^a) \,. \tag{2.24}$$

Interchanging k, l in (2.24) and subtracting the resulting equation from (2.24) we have

$$R_{akij}F_l{}^aF^{ij} = R_{alij}F_k{}^aF^{ij} \,, \tag{2.25}$$

and therefore (2.24) is reduced to

$$\nabla_k F_{ij}\nabla_l F^{ij} = R_{kl} - \tfrac{1}{2}R_{alij}F_k{}^aF^{ij} \,, \tag{2.26}$$

which together with (2.18) implies

$$\nabla_k F_{ij}\nabla_l F^{ij} = 0 \,. \tag{2.27}$$

Multiplying (2.27) by g^{kl} we hence obtain $\nabla_k F_{ij} = 0$.

Lemma 2.4. *For an almost-Hermitian structure F, condition*

$$F_i{}^kR_{jk} = F_k{}^lR^k{}_{ijl} \tag{2.28}$$

implies condition (2.18).

Proof. Since

$$\begin{aligned} F^{kl}R_{kjhl} &= \tfrac{1}{2}F^{kl}(R_{kjhl} - R_{ljhk}) \\ &= \tfrac{1}{2}F^{kl}(R_{kjhl} + R_{khlj}) = \tfrac{1}{2}F^{kl}R_{kljh} \end{aligned}$$

by the Bianchi identity, from (2.28) we obtain

$$F_j{}^k R_{hk} = \tfrac{1}{2} R_{jhkl} F^{kl} \,. \tag{2.29}$$

Multiplying (2.29) by $F_i{}^j$ and using (2.1) lead immediately to (2.18).

We now consider an almost-Hermitian manifold M^n with an almost-Hermitian structure F, and shall follow Spencer (compare [7, Chapter IX]) to introduce complex operators on the manifold M^n. At first we define

$$\prod_{1,0} {}_i{}^j = \tfrac{1}{2}(\varepsilon_i{}^j - \sqrt{-1} F_i{}^j) \tag{2.30}$$

and its conjugate[1] tensor

$$\prod_{0,1} {}_i{}^j = \overline{\prod_{1,0}} {}_i{}^j = \tfrac{1}{2}(\varepsilon_i{}^j + \sqrt{-1} F_i{}^j) \,. \tag{2.31}$$

A simple calculation gives the following identities:

$$\begin{aligned} \prod_{1,0} {}_i{}^j \prod_{1,0} {}_j{}^k &= \prod_{1,0} {}_i{}^k \,, \\ \prod_{1,0} {}_i{}^j \prod_{0,1} {}_j{}^k &= 0 \,, \\ \prod_{0,1} {}_i{}^j \prod_{0,1} {}_j{}^k &= \prod_{0,1} {}_i{}^k \,. \end{aligned} \tag{2.32}$$

Let $\rho + \sigma = p, \rho \geq 0, \sigma \geq 0$, set

$$\begin{aligned} \prod_{\rho,\sigma} {}_{I(p)}{}^{J(p)} = \varepsilon_{I(p)}^{M(\rho)N(\sigma)} \prod_{1,0} {}_{m_1}{}^{r_1} \cdots \prod_{1,0} {}_{m_\rho}{}^{r_\rho} \\ \cdot \prod_{0,1} {}_{n_1}{}^{s_1} \cdots \prod_{0,1} {}_{n_\sigma}{}^{s_\sigma} \varepsilon^{J(p)}_{R_0(\rho) S_0(\sigma)} \,, \end{aligned} \tag{2.33}$$

and define $\prod_{\rho,\sigma} {}_{I(p)}{}^{J(p)}$ to be the identity for $\rho = \sigma = 0$ and to be zero for either $\rho < 0$ or $\sigma < 0$. Then for a form ϕ given by (1.4) we have

$$\prod_{\rho,\sigma} \phi = (\prod_{\rho,\sigma} \phi)_{I_0(p)} dx^{I_0(p)} \,, \tag{2.34}$$

where

$$(\prod_{\rho,\sigma} \phi)_{I(p)} = \prod_{\rho,\sigma} {}_{I(p)}{}^{J_0(p)} \phi_{J_0(p)} \,. \tag{2.35}$$

We next define a complex covariant differentiator

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[1] Throughout this paper a bar over a letter or symbol denotes the conjugate of the complex number or operator defined by the letter or symbol.

$$\mathscr{D}_i = \prod_{1,0} {}_i{}^j\nabla_j \,, \tag{2.36}$$

and the corresponding contravariant differentiator

$$\mathscr{D}^i = g^{ik}\mathscr{D}_k = \prod_{0,1} {}_j{}^i\nabla^j = \overline{\prod_{1,0}} {}_j{}^i\nabla^j \,. \tag{2.37}$$

The conjugate operators of $\mathscr{D}_i$ and $\mathscr{D}^i$ are

$$\overline{\mathscr{D}}_i = \prod_{0,1} {}_i{}^j\nabla_j \,, \tag{2.38}$$

$$\overline{\mathscr{D}}^i = \prod_{1,0} {}_j{}^i\nabla^j \,. \tag{2.39}$$

Now we define the complex analogues of the real operators d and δ defined by (1.7), (1.15) respectively:

$$d_1 = \sum_{\rho+\sigma=p} \prod_{\rho+1,\sigma} d \prod_{\rho,\sigma} \,, \tag{2.40}$$

$$d_2 = \sum_{\rho+\sigma=p} \prod_{\rho+2,\sigma-1} d \prod_{\rho,\sigma} \,, \tag{2.41}$$

$$\delta_1 = \sum_{\rho+\sigma=p} \prod_{\rho,\sigma-1} \delta \prod_{\rho,\sigma} \,, \tag{2.42}$$

$$\delta_2 = \sum_{\rho+\sigma=p} \prod_{\rho+1,\sigma-2} \delta \prod_{\rho,\sigma} \,. \tag{2.43}$$

The conjugate operators of d_1, d_2 and δ_1, δ_2 have the forms:

$$\bar{d}_1 = \sum_{\rho+\sigma=p} \prod_{\rho,\sigma+1} d \prod_{\rho,\sigma} \,, \tag{2.44}$$

$$\bar{d}_2 = \sum_{\rho+\sigma=p} \prod_{\rho-1,\sigma+2} d \prod_{\rho,\sigma} \,, \tag{2.45}$$

$$\bar{\delta}_1 = \sum_{\rho+\sigma=p} \prod_{\rho-1,\sigma} \delta \prod_{\rho,\sigma} \,, \tag{2.46}$$

$$\bar{\delta}_2 = \sum_{\rho+\sigma=p} \prod_{\rho-2,\sigma+1} \delta \prod_{\rho,\sigma} \,. \tag{2.47}$$

Furthermore, for a p-form ϕ given by (1.4) we define

$$(\partial_1\phi)_{I(p+1)} = (2d_2 + d_1 - \bar{d}_2)_{I(p+1)} \,, \tag{2.48}$$

$$(\vartheta_1\phi)_{I(p-1)} = (2\delta_2 + \delta_1 - \bar{\delta}_2)_{I(p-1)} \,, \tag{2.49}$$

$$(\partial_2\phi)_{I(p+1)} = \sum_{\rho+\sigma=p} \prod_{\rho+1,\sigma} {}_{I(p+1)}{}^{jJ_0(p)} \mathscr{D}_j \phi_{J_0(p)} \,, \tag{2.50}$$

$$(\vartheta_2\phi)_{I(p-1)} = -\sum_{\rho+\sigma=p} \prod_{\rho,\sigma} {}_{iI(p-1)}{}^{J_0(p)} \mathscr{D}^i \phi_{J_0(p)} \,, \tag{2.51}$$

together with their conjugate operators:

$$(\partial_1\phi)_{I(p+1)} = (2\bar{d}_2 + \bar{d}_1 - d_2)_{I(p+1)} , \tag{2.52}$$

$$(\bar{\vartheta}_1\phi)_{I(p-1)} = (2\bar{\delta}_2 + \bar{\delta}_1 - \delta_2)_{I(p-1)} , \tag{2.53}$$

$$(\bar{\partial}_2\phi)_{I(p+1)} = \sum_{\rho+\sigma=p} \prod_{\rho,\sigma+1} {}_{I(p+1)}{}^{jJ_0(p)} \bar{\mathscr{D}}_j \phi_{J_0(p)} , \tag{2.54}$$

$$(\bar{\vartheta}_2\phi)_{I(p-1)} = - \sum_{\rho+\sigma=p} \prod_{\rho,\sigma} {}_{iI(p-1)}{}^{J_0(p)} \bar{\mathscr{D}}^i \phi_{J_0(p)} . \tag{2.55}$$

It is known that (see [3], [5])

$$\vartheta_1 = - * \partial_1 * , \qquad \vartheta_2 = - * \partial_2 * , \tag{2.56}$$

and that (see [3]) if the structure F of the manifold M^n is Kählerian, then $\bar{d}_2\phi = d_2\phi = 0$ for any form ϕ, and therefore $\partial_1 = d_1$.

Now we introduce the following complex Laplace-Beltrami operators:

$$\Box_i = \bar{\vartheta}_i\partial_i + \partial_i\bar{\vartheta}_i , \qquad (i = 1, 2) , \tag{2.57}$$

$$\Box_3 = \bar{\vartheta}_1\partial_2 + \partial_2\bar{\vartheta}_1 , \tag{2.58}$$

$$\Box_4 = \bar{\vartheta}_2\partial_1 + \partial_1\bar{\vartheta}_2 , \tag{5.59}$$

$$\Box_5 = \bar{\delta}_1 d_1 + d_1\bar{\delta}_1 . \tag{2.60}$$

It should be noted that $\Box_1$ was first defined by Kodaira-Spencer [3], and $\Box_2$ by Hsiung [2].

From [3] we know that $d = \partial_1 + \bar{\partial}_1$. In order to apply $\partial_2 + \bar{\partial}_2$, let ξ be any 0-form. The we have, in consequence of (2.50), (2.36), (2.32), (2.30),

$$(\partial_2\xi)_{i_1} = \prod_{1,0} {}_{i_1}{}^j \nabla_j \xi = \tfrac{1}{2}(\nabla_{i_1}\xi - \sqrt{-1}F_{i_1}{}^j\nabla_j\xi) , \tag{2.61}$$

which together with (1.6), (1.7) gives

$$d\xi = (\partial_2 + \bar{\partial}_2)\xi . \tag{2.62}$$

Similarly, for any 1-form η, using (2.50), (2.36), (2.33), (2.34), (2.35), (2.32) we can obtain

$$\begin{aligned}(\partial_2\eta)_{i_1 i_2} &= \prod_{1,0} {}_{i_1}{}^j \prod_{1,0} {}_{i_2}{}^k (\nabla_j\eta_k - \nabla_k\eta_j) + (\prod_{1,0} {}_{i_1}{}^j \prod_{0,1} {}_{i_2}{}^k - \prod_{1,0} {}_{i_2}{}^j \prod_{0,1} {}_{i_1}{}^k)\nabla_j\eta_k \\ &= \frac{1}{2}[\nabla_{i_1}\eta_{i_2} - \nabla_{i_2}\eta_{i_1} + \sqrt{-1}(F_{i_2}{}^j\nabla_j\eta_{i_1} - F_{i_1}{}^j\nabla_j\eta_{i_2})] ,\end{aligned} \tag{2.63}$$

which together with (1.6), (1.7) gives

$$d\eta = \tfrac{1}{2}(\partial_2 + \bar{\partial}_2)\eta . \tag{2.64}$$

The almost-complex structure F of the manifold M^n is said [3] to be (completely) integrable if and only if $\partial_1^2 = 0$. Now by means of (2.61), (2.50), (2.30), $\cdots$, (2.36) and the relation

$$\nabla_i \nabla_j \xi = \nabla_j \nabla_i \xi \tag{2.65}$$

for any 0-form ξ, an elementary but lengthy calculation gives

$$\begin{aligned} 4(\partial_2^2 \xi)_{i_1 i_2} = {} & (F_{i_2}{}^j \nabla_j F_{i_1}{}^k - F_{i_1}{}^j \nabla_j F_{i_2}{}^k) \nabla_k \xi \\ & + \sqrt{-1}(\nabla_{i_2} F_{i_1}{}^k - \nabla_{i_1} F_{i_2}{}^k) \nabla_k \xi \, . \end{aligned} \tag{2.66}$$

If ∂_2^2 *is real for any* 0-*form* ξ, then by taking $\xi = x^i$ for any arbitary i with respect to any local coordinates $x^1, \cdots, x^n$, from (2.66) we obtain (2.14), and therefore by Lemma 2.1 *the structure F is Kählerian.*

3. Expressions for $\Box$'s

In this section we shall give expressions for $\Box_i \xi$ and $\Box_i \eta$, where $i = 1, \cdots, 4$, and ξ and η are respectively any 0- and 1-forms on an almost-Hermitian manifold M^n with an almost-Hermitian structure F.

3.1. Laplacian $\Box_2$. In [2, pp. 146–147] we obtained

$$4\Box_2 \xi = 2\Delta\xi + \nabla^h F_h{}^j(-F_j{}^k \nabla_k \xi + \sqrt{-1}\nabla_j \xi) \, , \tag{3.1}$$

$$\begin{aligned} 4(\Box_2 \eta)_{i_1} = {} & -F_i{}^j \nabla^i F_{i_1}{}^h \nabla_h \eta_j - F_j{}^h \nabla^i F_i{}^j \nabla_h \eta_{i_1} + F_{i_1}{}^i \nabla_i F_h{}^j \nabla^h \eta_j \\ & - 2\nabla^i \nabla_i \eta_{i_1} + [\nabla_j, \nabla_{i_1}]\eta^j + F_{i_1}{}^h F^{ij}[\nabla_h, \nabla_i]\eta_j \\ & + \sqrt{-1}\{\nabla^i F_i{}^j \nabla_j \eta_{i_1} - (\nabla_j F_{i_1}{}^k + \nabla_{i_1} F_j{}^k)\nabla_k \eta^j \\ & + 2F_j{}^k \nabla^j \nabla_k \eta_{i_1} - F_{i_1}{}^k [\nabla_j, \nabla_k]\eta^j - F^{kj}[\nabla_k, \nabla_{i_1}]\eta_j\} \, , \end{aligned} \tag{3.2}$$

where

$$[\nabla_h, \nabla_i] = \nabla_h \nabla_i - \nabla_i \nabla_h \, . \tag{3.3}$$

3.2. Laplacian $\Box_1$. At first we notice that as a result of (2.65) we have

$$F_i{}^j \nabla^i \nabla_j \xi = 0 \, . \tag{3.4}$$

By using (2.57), (2.53), (2.45), $\cdots$, (2.48), (2.33), ((2.40), (2.41), (2.43), (2.1), (2.30), (2.32), (2.34), (2.35), (3.4), (1.17), (1.18) we can obtain

$$2\Box_1 \xi = 2\delta \prod_{1,0} d\xi = \Delta\xi + \sqrt{-1}\nabla^i F_i{}^j \nabla_j \xi \, . \tag{3.5}$$

In order to compute $\Box_1 \eta$, from (2.48), (2.52), (2.40), (2.41), (2.43), (2.45), (2.46), (2.47) we first see that

$$\partial_1 = 2 \prod_{2,0} d \prod_{0,1} + \prod_{2,0} d \prod_{1,0} + \prod_{1,1} d \prod_{0,1} - \prod_{0,2} d \prod_{1,0} \, , \qquad \text{for 1-forms} \, , \tag{3.6}$$

$$\overline{\vartheta}_1 = 2 \prod_{0,1} \delta \prod_{2,0} + \prod_{1,0} \delta \prod_{2,0} + \prod_{0,1} \delta \prod_{1,1} - \prod_{1,0} \delta \prod_{0,2} , \qquad \text{for 2-forms} . \tag{3.7}$$

Next, by means of (1.6), (1.7), (2.33), (2.34), (2.35), (1.2), (1.3), (2.30), (2.31), we obtain

$$\begin{aligned}
\prod_{1,1} d \prod_{0,1} \eta = {} & [\prod_{1,0}{}_{i_1}{}^{k} \prod_{1,0}{}_{i_2}{}^{l}(\nabla_k \prod_{0,1}{}_{l}{}^{j} - \nabla_l \prod_{0,1}{}_{k}{}^{j})\eta_j + \prod_{1,0}{}_{i_1}{}^{k} \prod_{0,1}{}_{i_2}{}^{j}\nabla_k \eta_j \\
& - \prod_{1,0}{}_{i_2}{}^{k} \prod_{0,1}{}_{i_1}{}^{j}\nabla_k \eta_j + \prod_{1,0}{}_{i_2}{}^{k} \prod_{0,1}{}_{i_1}{}^{l}(\nabla_l \prod_{0,1}{}_{k}{}^{j} - \nabla_k \prod_{0,1}{}_{l}{}^{j})\eta_j] dx^{I_0(2)} \\
= {} & \tfrac{1}{4}\{\nabla_{i_1}\eta_{i_2} - \nabla_{i_2}\eta_{i_1} + F_{i_1}{}^{k} F_{i_2}{}^{j}(\nabla_k \eta_j - \nabla_j \eta_k) \\
& + \sqrt{-1}[\eta_j(\nabla_{i_1} F_{i_2}{}^{j} - \nabla_{i_2} F_{i_1}{}^{j} + F_{i_1}{}^{k} F_{i_2}{}^{l}(\nabla_k F_l{}^{j} - \nabla_l F_k{}^{j})) \\
& + F_{i_2}{}^{j}(\nabla_{i_1}\eta_j + \nabla_j \eta_{i_1}) - F_{i_1}{}^{j}(\nabla_{i_2}\eta_j + \nabla_j \eta_{i_2})]\} dx^{I_0(2)} ,
\end{aligned} \tag{3.8}$$

$$\begin{aligned}
\prod_{2,0} d \prod_{1,0} \eta = {} & [\prod_{1,0}{}_{i_1}{}^{k} \prod_{1,0}{}_{i_2}{}^{l}(\nabla_k \prod_{1,0}{}_{l}{}^{j} - \nabla_l \prod_{1,0}{}_{k}{}^{j})\eta_j \\
& + \prod_{1,0}{}_{i_1}{}^{j} \prod_{1,0}{}_{i_2}{}^{k}(\nabla_j \eta_k - \nabla_k \eta_j)] dx^{I_0(2)} \\
= {} & \tfrac{1}{8}\{\eta_j[F_{i_1}{}^{k}(\nabla_{i_2} F_k{}^{j} - \nabla_k F_{i_2}{}^{j}) + F_{i_2}{}^{k}(\nabla_k F_{i_1}{}^{j} - \nabla_{i_1} F_k{}^{j})] \\
& + 2\nabla_{i_1}\eta_{i_2} - 2\nabla_{i_2}\eta_{i_1} + 2F_{i_1}{}^{j} F_{i_2}{}^{k}(\nabla_k \eta_j - \nabla_j \eta_k) \\
& + \sqrt{-1}[\eta_j(\nabla_{i_2} F_{i_1}{}^{j} - \nabla_{i_1} F_{i_2}{}^{j} + F_{i_1}{}^{k} F_{i_2}{}^{l}(\nabla_k F_l{}^{j} - \nabla_l F_k{}^{j})) \\
& + 2F_{i_1}{}^{j}(\nabla_{i_2}\eta_j - \nabla_j \eta_{i_2}) + 2F_{i_2}{}^{j}(\nabla_j \eta_{i_1} - \nabla_{i_1}\eta_j)]\} dx^{I_0(2)} ,
\end{aligned} \tag{3.9}$$

$$\begin{aligned}
\prod_{2,0} d \prod_{0,1} \eta = {} & \prod_{1,0}{}_{i_1}{}^{k} \prod_{1,0}{}_{i_2}{}^{l}(\nabla_k \prod_{0,1}{}_{l}{}^{j} - \nabla_l \prod_{0,1}{}_{k}{}^{j})\eta_j dx^{I_0(2)} \\
= {} & \tfrac{1}{8}\{\eta_j[F_{i_1}{}^{k}(\nabla_k F_{i_2}{}^{j} - \nabla_{i_2} F_k{}^{j}) + F_{i_2}{}^{k}(\nabla_{i_1} F_k{}^{j} - \nabla_k F_{i_1}{}^{j})] \\
& + \sqrt{-1}\eta_j[\nabla_{i_1} F_{i_2}{}^{j} - \nabla_{i_2} F_{i_1}{}^{j} \\
& + F_{i_1}{}^{k} F_{i_2}{}^{l}(\nabla_l F_k{}^{j} - \nabla_k F_l{}^{j})]\} dx^{I_0(2)} ,
\end{aligned} \tag{3.10}$$

$$\begin{aligned}
- \prod_{0,2} d \prod_{1,0} \eta = {} & \prod_{0,1}{}_{i_2}{}^{k} \prod_{0,1}{}_{i_1}{}^{l}(\nabla_k \prod_{1,0}{}_{l}{}^{j} - \nabla_l \prod_{1,0}{}_{k}{}^{j})\eta_j dx^{I_0(2)} \\
= {} & \tfrac{1}{8}\{\eta_j[F_{i_1}{}^{k}(\nabla_{i_2} F_k{}^{j} - \nabla_k F_{i_2}{}^{j}) + F_{i_2}{}^{k}(\nabla_k F_{i_1}{}^{j} - \nabla_{i_1} F_k{}^{j})] \\
& + \sqrt{-1}\eta_j[\nabla_{i_1} F_{i_2}{}^{j} - \nabla_{i_2} F_{i_1}{}^{j} \\
& + F_{i_1}{}^{k} F_{i_2}{}^{l}(\nabla_l F_k{}^{j} - \nabla_k F_l{}^{j})]\} dx^{I_0(2)} ,
\end{aligned} \tag{3.11}$$

$$\begin{aligned}
4(\prod_{1,0} d\delta \prod_{1,0} \eta)_{i_1} = {} & \eta_j F_{i_1}{}^{l} \nabla_l \nabla^k F_k{}^{j} + F_{i_1}{}^{l} \nabla^k F_k{}^{j} \nabla_l \eta_j \\
& + F_{i_1}{}^{l} \nabla_l F_k{}^{j} \nabla^k \eta_j - \nabla_{i_1} \nabla^k \eta_k + F_{i_1}{}^{l} F_k{}^{j} \nabla_l \nabla^k \eta_j \\
& + \sqrt{-1}(\eta_j \nabla_{i_1} \nabla^k F_k{}^{j} + \nabla^k F_k{}^{j} \nabla_{i_1} \eta_j + \nabla_{i_1} F_k{}^{j} \nabla^k \eta_j \\
& + F_k{}^{j} \nabla_{i_1} \nabla^k \eta_j + F_{i_1}{}^{l} \nabla_l \nabla^k \eta_k) .
\end{aligned} \tag{3.12}$$

Substitution of (3.8), $\cdots$, (3.11) in (3.6) thus gives

$$\partial_1\eta = \tfrac{1}{2}\{\nabla_{i_1}\eta_{i_2} - \nabla_{i_2}\eta_{i_1} + \sqrt{-1}[\eta_j(\nabla_{i_1}F_{i_2}{}^j - \nabla_{i_2}F_{i_1}{}^j) + F_{i_2}{}^j\nabla_j\eta_{i_1} - F_{i_1}{}^j\nabla_j\eta_{i_2}]\}dx^{I_{0(2)}} . \tag{3.13}$$

Now put

$$\begin{aligned} A_{i_1i_2} &= \nabla_{i_1}\eta_{i_2} + \sqrt{-1}(\eta_j\nabla_{i_1}F_{i_2}{}^j + F_{i_2}{}^j\nabla_j\eta_{i_1}) , \\ B^{k_1k_2}_{si_1} &= \varepsilon^{k_1}_s\varepsilon^{k_2}_{i_1} - \varepsilon^{k_1}_{i_1}\varepsilon^{k_2}_s + F_{i_1}{}^{k_1}F_s{}^{k_2} - F_s{}^{k_1}F_{i_1}{}^{k_2} , \\ C^{k_1k_2}_{si_1} &= \varepsilon^{k_1}_{i_1}F_s{}^{k_2} - \varepsilon^{k_1}_sF_{i_1}{}^{k_2} - \varepsilon^{k_2}_{i_1}F_s{}^{k_1} + \varepsilon_s{}^{k_2}F_{i_1}{}^{k_1} . \end{aligned} \tag{3.14}$$

Then

$$\partial_1\eta = \tfrac{1}{2}(A_{i_1i_2} - A_{i_2i_1})dx^{I_{0(2)}} . \tag{3.15}$$

By means of (3.13), (2.33), (2.34), (2.35), (2.30), (2.31), (1.16), (1.17), elementary but rather lengthy calculations give

$$\begin{aligned} &-8(\prod_{0,1}\delta\prod_{1,1}\partial_1\eta)_{i_1} \\ &= 2\prod_{0,1}\nabla^s[\varepsilon^{k_1}_s\varepsilon^{k_2}_{i_1} - \varepsilon^{k_1}_{i_1}\varepsilon^{k_2}_s + F_s{}^{k_1}F_{i_1}{}^{k_2} - F_{i_1}{}^{k_1}F_s{}^{k_2})A_{k_1k_2}] \\ &= \eta_j[\nabla^sF_s{}^k(\nabla_kF_{i_1}{}^j - \nabla_{i_1}F_k{}^j) + F_{i_1}{}^rF_s{}^k\nabla^sF_r{}^l(\nabla_lF_k{}^j - \nabla_kF_l{}^j) \\ &\qquad + F_{i_1}{}^r(\nabla^s\nabla_rF_s{}^j - \nabla^s\nabla_sF_r{}^j) + F_s{}^k(\nabla^s\nabla_kF_{i_1}{}^j - \nabla^s\nabla_{i_1}F_k{}^j)] \\ &\quad + (2F_{i_1}{}^l\nabla^sF_s{}^k - F_s{}^l\nabla^sF_{i_1}{}^k)\nabla_k\eta_l + F_{i_1}{}^r\nabla^s\eta_j(\nabla_rF_s{}^j - 2\nabla_sF_r{}^j) \\ &\quad - F_s{}^k\nabla_{i_1}F_k{}^j\nabla^s\eta_j - F_{i_1}{}^r\nabla^sF_r{}^j\nabla_j\eta_s + 2\nabla^s\nabla_s\eta_{i_1} \\ &\quad + 2F_{i_1}{}^kF_s{}^l\nabla^s\nabla_l\eta_k + \sqrt{-1}\{\eta_j[F_s{}^k\nabla^sF_{i_1}{}^l(\nabla_kF_l{}^j - \nabla_lF_k{}^j) \\ &\quad + F_{i_1}{}^k\nabla^sF_s{}^l(\nabla_lF_k{}^j - \nabla_kF_l{}^j) + \nabla^s\nabla_sF_{i_1}{}^j - \nabla^s\nabla_{i_1}F_s{}^j \\ &\quad + F_{i_1}{}^kF_s{}^l(\nabla^s\nabla_lF_k{}^j - \nabla^s\nabla_kF_l{}^j)] + (2\nabla_sF_{i_1}{}^j - \nabla_{i_1}F_s{}^j)\nabla^s\eta_j \\ &\quad - 2\nabla^sF_s{}^j\nabla_j\eta_{i_1} + \nabla^sF_{i_1}{}^j\nabla_j\eta_s - F_{i_1}{}^kF_s{}^l(\nabla_kF_l{}^j\nabla^s\eta_j + \nabla^sF_k{}^j\nabla_j\eta_l) \\ &\quad + 2F_{i_1}{}^j\nabla^s\nabla_s\eta_j - 2F_s{}^j\nabla^s\nabla_j\eta_{i_1}\} , \end{aligned} \tag{3.16}$$

$$\begin{aligned} &-8(\prod_{1,0}\delta\prod_{2,0}\partial_1\eta)_{i_1} \\ &= \prod_{1,0}\nabla^s[(B^{k_1k_2}_{s\,i_1} + \sqrt{-1}C^{k_1k_2}_{s\,i_1})A_{k_1k_2}] \\ &= \eta_j[\tfrac{1}{2}\nabla^sF_{i_1}{}^k(\nabla_sF_k{}^j - \nabla_kF_s{}^j) + \tfrac{1}{2}F_{i_1}{}^rF_s{}^k\nabla^sF_r{}^l(\nabla_lF_k{}^j - \nabla_kF_l{}^j) \\ &\qquad + \nabla^sF_s{}^k(\nabla_kF_{i_1}{}^j - \nabla_{i_1}F_k{}^j) + F_{i_1}{}^k(\nabla^s\nabla_sF_k{}^j - \nabla^s\nabla_kF_s{}^j) \\ &\qquad + F_s{}^k(\nabla^s\nabla_kF_{i_1}{}^j - \nabla^s\nabla_{i_1}F_k{}^j)] + 2F_{i_1}{}^k\nabla^sF_s{}^l(\nabla_k\eta_l - \nabla_l\eta_k) \end{aligned}$$

$$
\begin{aligned}
&+ F_{i_1}{}^r \nabla_j \eta^s (\nabla_r F_s{}^j + \nabla_s F_r{}^j) + F_s{}^k \nabla^s F_{i_1}{}^l (\nabla_l \eta_k - 2\nabla_k \eta_l) \\
&- F_s{}^k \nabla_{i_1} F_k{}^j \nabla^s \eta_j + 2\nabla^s \nabla_s \eta_{i_1} - 2\nabla^s \nabla_{i_1} \eta_s \\
&+ 2F_{i_1}{}^k F_s{}^l (\nabla^s \nabla_k \eta_l - \nabla^s \nabla_l \eta_k) \\
&+ \sqrt{-1}\{\eta_j[\tfrac{1}{2} F_{i_1}{}^r \nabla^s F_r{}^k (\nabla_k F_s{}^j - \nabla_s F_k{}^j) + F_{i_1}{}^k \nabla^s F_s{}^l (\nabla_k F_l{}^j \\
&- \nabla_l F_k{}^j) + \tfrac{1}{2} F_s{}^k \nabla^s F_{i_1}{}^l (\nabla_l F_k{}^j - \nabla_k F_l{}^j) + \nabla^s \nabla_s F_{i_1}{}^j \\
&- \nabla^s \nabla_{i_1} F_s{}^j + F_{i_1}{}^k F_s{}^l (\nabla^s \nabla_k F_l{}^j - \nabla^s \nabla_l F_k{}^j)] + \nabla^s F_{i_1}{}^j \nabla_j \eta_s \\
&- \nabla_{i_1} F_s{}^j \nabla^s \eta_j + F_{i_1}{}^k F_s{}^l \nabla_k F_l{}^j \nabla^s \eta_j - F_{i_1}{}^r F_s{}^l \nabla^s F_r{}^k \nabla_k \eta_l \\
&+ 2\nabla^s F_s{}^j (\nabla_{i_1} \eta_j - \nabla_j \eta_{i_1}) - 2F_s{}^k F_{i_1}{}^l \nabla_k F_l{}^j \nabla^s \eta_j \\
&+ 2F_s{}^k (\nabla^s \nabla_{i_1} \eta_k - \nabla^s \nabla_k \eta_{i_1}) + 2F_{i_1}{}^j (\nabla^s \nabla_j \eta_s - \nabla^s \nabla_s \eta_j)\} ,
\end{aligned}
\tag{3.17}
$$

$$
\begin{aligned}
-8&(\prod_{0,1} \delta \prod_{2,0} \partial_1 \eta)_{i_1} \\
&= \eta_j[\tfrac{1}{2} \nabla^s F_{i_1}{}^k (\nabla_s F_k{}^j - \nabla_k F_s{}^j) + \tfrac{1}{2} F_{i_1}{}^r F_s{}^k \nabla^s F_r{}^l (\nabla_k F_l{}^j - \nabla_l F_k{}^j)] \\
&\quad + F_s{}^k \nabla^s F_{i_1}{}^l (\nabla_l \eta_k - \nabla_k \eta_l) + F_{i_1}{}^r \nabla^s F_r{}^k (\nabla_s \eta_k - \nabla_k \eta_s) \\
&\quad + \sqrt{-1}\{\eta_j[\tfrac{1}{2} F_s{}^k \nabla^s F_{i_1}{}^l (\nabla_l F_k{}^j - \nabla_k F_l{}^j) \\
&\qquad\qquad + \tfrac{1}{2} F_{i_1}{}^r \nabla^s F_r{}^k (\nabla_s F_k{}^j - \nabla_k F_s{}^j)] \\
&\qquad\qquad + F_{i_1}{}^r F_s{}^k \nabla^s F_r{}^l (\nabla_l \eta_k - \nabla_k \eta_l) + \nabla^s F_{i_1}{}^j (\nabla_j \eta_s - \nabla_s \eta_j)\} ,
\end{aligned}
\tag{3.18}
$$

$$
\begin{aligned}
-16&(\prod_{1,0} \delta \prod_{0,2} \partial_1 \eta)_{i_1} = 2 \prod_{1,0} \nabla^s [(B^{k_1 k_2}_{s\ i_1} - \sqrt{-1} C^{k_1 k_2}_{s\ i_1}) A_{k_1 k_2}] \\
&= \eta_j[\nabla^s F_{i_1}{}^k (\nabla_k F_s{}^j - \nabla_s F_k{}^j) + F_{i_1}{}^r F_s{}^k \nabla^s F_r{}^l (\nabla_l F_k{}^j - \nabla_k F_l{}^j)] \\
&\quad + \sqrt{-1}\eta_j[F_s{}^k \nabla^s F_{i_1}{}^l (\nabla_l F_k{}^j - \nabla_k F_l{}^j) \\
&\qquad\qquad + F_{i_1}{}^r \nabla^s F_r{}^k (\nabla_s F_k{}^j - \nabla_k F_s{}^j)] .
\end{aligned}
\tag{3.19}
$$

Substituting (3.6), (3.7), (3.12), (3.16), $\cdots$, (3.19) in (2.57) and using (2.32) and

$$
2F_j{}^k \nabla^j \nabla_k \eta_{i_1} = F^{jk} [\nabla_j, \nabla_k] \eta_{i_1} , \tag{3.20}
$$

we can obtain, after some elementary simplification,

$$
\begin{aligned}
4(\Box_1 \eta)_{i_1} &= 4[(2 \prod_{0,1} \delta \prod_{2,0} + \prod_{1,0} \delta \prod_{2,0} + \prod_{0,1} \delta \prod_{1,1} - \prod_{1,0} \delta \prod_{0,2}) \partial_1 \eta + \prod_{1,0} d\delta \prod_{1,0} \eta]_{i_1} \\
&= \eta_j[\nabla^s F_s{}^k (\nabla_{i_1} F_k{}^j - \nabla_k F_{i_1}{}^j) + \nabla^s F_{i_1}{}^k (\nabla_k F_s{}^j - \nabla_s F_k{}^j) \\
&\qquad + F_s{}^k (\nabla^s \nabla_{i_1} F_k{}^j - \nabla^s \nabla_k F_{i_1}{}^j) + F_{i_1}{}^l \nabla_l \nabla^k F_k{}^j] \\
&\quad + F_s{}^k \nabla^s \eta_j (\nabla_{i_1} F_k{}^j - 2\nabla_k F_{i_1}{}^j) + F_s{}^k \nabla^s F_{i_1}{}^l (\nabla_k \eta_l - \nabla_l \eta_k) \\
&\quad + F_{i_1}{}^l \nabla_l F_k{}^j \nabla^k \eta_j + F_{i_1}{}^l \nabla^j F_l{}^k \nabla_k \eta_j - 2\nabla^s \nabla_s \eta_{i_1} \\
&\quad + [\nabla_k, \nabla_{i_1}] \eta^k + F_{i_1}{}^l F^{kj} [\nabla_l, \nabla_k] \eta_j \\
&\quad + \sqrt{-1}\{\eta_j (\nabla^s \nabla_{i_1} F_s{}^j - \nabla^s \nabla_s F_{i_1}{}^j + \nabla_{i_1} \nabla^s F_s{}^j) + 2\nabla^s F_s{}^j \nabla_j \eta_{i_1}
\end{aligned}
\tag{3.21}
$$

$$- 2(\nabla_s F_{i_1}{}^j + \nabla_{i_1} F_s{}^j)\nabla_j \eta^s + F^{sj}[\nabla_s, \nabla_j]\eta_{i_1} - F^{sj}[\nabla_s, \nabla_{i_1}]\eta_j - F_{i_1}{}^j[\nabla_s, \nabla_j]\eta^s\} \,.$$

3.3. Laplacian $\square_3$. In the same way as above we can compute $\square_i \xi$ and $\square_i \eta$, $i = 3, 4, 5$, but we shall omit the details in this section and §§ 3.4, 3.5. We find that

$$\square_3 \xi = \square_1 \xi \,, \tag{3.22}$$

$$2\bar{\vartheta}_1 \eta = -\nabla^k \eta_k + \sqrt{-1}(F_k{}^j \nabla^k \eta_j + \eta_j \nabla^k F_k{}^j) \,, \tag{3.23}$$

$$\begin{aligned} 4(\partial_2 \bar{\vartheta}_1 \eta)_{i_1} = {} & \eta_j F_{i_1}{}^l \nabla_l \nabla^k F_k{}^j + F_{i_1}{}^l \nabla_k{}^j \nabla^k \eta_j + F_{i_1}{}^l \nabla^k F_k{}^j \nabla_l \eta_j \\ & - \nabla_i \nabla^k \eta_k + F_{i_1}{}^l F_k{}^j \nabla_l \nabla^k \eta_j + \sqrt{-1}[\eta_j \nabla_{i_1} \nabla^k F_k{}^j \\ & + \nabla_{i_1} F_k{}^j \nabla^k \eta_j + \nabla^k F_k{}^j \nabla_{i_1} \eta_j + F_k{}^j \nabla_{i_1} \nabla^k \eta_j + F_{i_1}{}^l \nabla_l \nabla^k \eta_k] \,, \end{aligned} \tag{3.24}$$

$$\begin{aligned} 2(\bar{\vartheta}_1 \partial_2 \eta)_{i_1} = {} & 2[(2 \prod_{0,1} \delta \prod_{2,0} + \prod_{1,0} \delta \prod_{2,0} + \prod_{0,1} \delta \prod_{1,1} - \prod_{1,0} \delta \prod_{0,2}) \partial_2 \eta]_{i_1} \\ = {} & F_s{}^j \nabla^s F_{i_1}{}^k (\nabla_j \eta_k - \nabla_k \eta_j) + F_{i_1}{}^r (\nabla^s F_r{}^j \nabla_j \eta_s - \nabla^s F_s{}^j \nabla_r \eta_j) \\ & - 2\nabla^s \nabla_s \eta_{i_1} + \nabla^s \nabla_{i_1} \eta_s - F_{i_1}{}^k F_s{}^l \nabla^s \nabla_k \eta_l \\ & + \sqrt{-1}[\nabla^s F_{i_1}{}^j (\nabla_s \eta_j - 2\nabla_j \eta_s) + \nabla^s F_s{}^j (2\nabla_j \eta_{i_1} - \nabla_{i_1} \eta_j) \\ & \quad - F_{i_1}{}^j \nabla^s \nabla_j \eta_s + F_s{}^j (2\nabla^s \nabla_j \eta_{i_1} - \nabla^s \nabla_{i_1} \eta_j)] \,, \end{aligned} \tag{3.25}$$

$$\begin{aligned} 4(\square_3 \eta)_{i_1} = {} & \eta_j F_{i_1}{}^l \nabla_l \nabla^k F_k{}^j + 2F_s{}^j \nabla^s F_{i_1}{}^k (\nabla_j \eta_k - \nabla_k \eta_j) \\ & + F_{i_1}{}^r (\nabla_r F_k{}^j \nabla^k \eta_j - \nabla^k F_k{}^j \nabla_r \eta_j) + 2F_{i_1}{}^r \nabla^s F_r{}^j \nabla_j \eta_s \\ & - 4\nabla^s \nabla_s \eta_{i_1} + 2\nabla^s \nabla_{i_1} \eta_s - \nabla_{i_1} \nabla^k \eta_k \\ & + F_{i_1}{}^l F_k{}^j (\nabla_l \nabla^k \eta_j - 2\nabla^k \nabla_l \eta_j) \\ & + \sqrt{-1}[\eta_j \nabla_{i_1} \nabla^k F_k{}^j + \nabla^s F_s{}^j (4\nabla_j \eta_{i_1} - \nabla_{i_1} \eta_j) \\ & \quad + 2\nabla^s F_{i_1}{}^j (\nabla_s \eta_j - 2\nabla_j \eta_s) \\ & \quad + \nabla_{i_1} F_k{}^j \nabla^k \eta_j + 2F_s{}^j (2\nabla^s \nabla_j \eta_{i_1} - \nabla^s \nabla_{i_1} \eta_i) \\ & \quad + F_{i_1}{}^j (\nabla_j \nabla^k \eta_k - 2\nabla^k \nabla_j \eta_k) + F_k{}^j \nabla_{i_1} \nabla^k \eta_j] \,. \end{aligned} \tag{3.26}$$

3.4. Laplacian $\square_4$. For $\square_4 \xi$, $\square_4 \eta$ we obtain the following equations:

$$2(\partial_1 \xi)_{i_1} = \nabla_{i_1} \xi - \sqrt{-1} F_{i_1}{}^j \nabla_j \xi \,, \tag{3.27}$$

$$\square_4 \xi = \square_2 \xi \,, \tag{3.28}$$

$$2\bar{\vartheta}_2 \eta = -\nabla^k \eta_k + \sqrt{-1} F_k{}^j \nabla^k \eta_j \,, \tag{3.29}$$

$$\begin{aligned} 4(\partial_1 \bar{\vartheta}_2 \eta)_{i_1} = {} & F_{i_1}{}^l \nabla_l F_k{}^j \nabla^k \eta_j - \nabla_{i_1} \nabla^k \eta_k + F_{i_1}{}^l F_k{}^j \nabla_l \nabla^k \eta_j \\ & + \sqrt{-1}(\nabla_{i_1} F_k{}^j \nabla^k \eta_j + F_k{}^j \nabla_{i_1} \nabla^k \eta_j + F_{i_1}{}^j \nabla_j \nabla^k \eta_k) \,, \end{aligned} \tag{3.30}$$

$$
\begin{aligned}
-4(\bar{\vartheta}_2\partial_1\eta)_{i_1} = {} & \eta_j F_k{}^l(\nabla^k\nabla_l F_{i_1}{}^j - \nabla^k\nabla_{i_1}F_l{}^j) + (\nabla_l F_{i_1}{}^j - \nabla_{i_1}F_l{}^j)F_k{}^l\nabla^k\eta_j \\
& + F_l{}^j(\nabla^l F_{i_1}{}^k\nabla_k\eta_j - \nabla^l F_j{}^k\nabla_k\eta_{i_1}) + 2\nabla^j\nabla_j\eta_{i_1} - \nabla^j\nabla_{i_1}\eta_j \\
& + F_l{}^j F_{i_1}{}^k\nabla^l\nabla_k\eta_j + \sqrt{-1}[(\nabla^j\nabla_j F_{i_1}{}^k - \nabla^j\nabla_{i_1}F_j{}^k)\eta_k \\
& + (\nabla_k F_{i_1}{}^j - \nabla_{i_1}F_k{}^j)\nabla^k\eta_j + \nabla^j F_{i_1}{}^k\nabla_k\eta_j - \nabla^j F_j{}^k\nabla_k\eta_{i_1} \\
& + F_{i_1}{}^j\nabla^k\nabla_j\eta_k - 2F_j{}^k\nabla^j\nabla_k\eta_{i_1} + F_j{}^k\nabla^j\nabla_{i_1}\eta_k] ,
\end{aligned} \tag{3.31}
$$

$$
\begin{aligned}
-4(\Box_4\eta)_{i_1} = {} & \eta_j F_k{}^l(\nabla^k\nabla_l F_{i_1}{}^j - \nabla^k\nabla_{i_1}F_l{}^j) + (\nabla_l F_{i_1}{}^j - \nabla_{i_1}F_l{}^j)F_k{}^l\nabla^k\eta_j \\
& + F_l{}^j(\nabla^l F_{i_1}{}^k\nabla_k\eta_j - \nabla^l F_j{}^k\nabla_k\eta_{i_1}) - F_{i_1}{}^l\nabla_l F_k{}^j\nabla^k\eta_j \\
& + [\nabla_{i_1}, \nabla_j]\eta^j + 2\nabla^j\nabla_j\eta_{i_1} + F_l{}^j F_{i_1}{}^k\nabla^l\nabla_k\eta_j \\
& - F_{i_1}{}^l F_k{}^j\nabla_l\nabla^k\eta_j \\
& + \sqrt{-1}\{(\nabla^j\nabla_j F_{i_1}{}^k - \nabla^j\nabla_{i_1}F_j{}^k)\eta_k \\
& \qquad + (\nabla_k F_{i_1}{}^j - 2\nabla_{i_1}F_k{}^j)\nabla^k\eta_j \\
& \qquad + \nabla^j F_{i_1}{}^k\nabla_k\eta_j - \nabla^j F_j{}^k\nabla_k\eta_{i_1} + F^{kj}[\nabla_k, \nabla_{i_1}]\eta_j \\
& \qquad + F^{kj}[\nabla_j, \nabla_k]\eta_{i_1} + F_{i_1}{}^j[\nabla_k, \nabla_j]\eta^k\} .
\end{aligned} \tag{3.32}
$$

3.5. Laplacian $\Box_5$. Finally, for the remaining Laplacian $\Box_5$ we first have

$$\Box_5\xi = \Box_1\xi , \tag{3.33}$$

$$\bar{\delta}_1\eta = -\eta_j\nabla^i \prod_{1,0} {}_i{}^j - \prod_{1,0} {}_i{}^j\nabla^i\eta_j . \tag{3.34}$$

Adding (3.8) to (3.9) gives

$$
\begin{aligned}
8(d_1\eta)_{i_1 i_2} = {} & 4\nabla_{i_1}\eta_{i_2} - 4\nabla_{i_2}\eta_{i_1} + \eta_j[F_{i_1}{}^k(\nabla_{i_2}F_k{}^j - \nabla_k F_{i_2}{}^j) \\
& \qquad + F_{i_2}{}^k(\nabla_k F_{i_1}{}^j - \nabla_{i_1}F_k{}^j)] \\
& + \sqrt{-1}\{\eta_j[3F_{i_1}{}^k F_{i_2}{}^l(\nabla_k F_l{}^j - \nabla_l F_k{}^j) \\
& \qquad + \nabla_{i_1}F_{i_2}{}^j - \nabla_{i_2}F_{i_1}{}^j] + 4F_{i_2}{}^k\nabla_k\eta_{i_1} \\
& \qquad - 4F_{i_1}{}^k\nabla_k\eta_{i_2}\} .
\end{aligned} \tag{3.35}
$$

Now put

$$
\begin{aligned}
8G_{i_1 i_2} = {} & 4\nabla_{i_1}\eta_{i_2} + \eta_j(F_{i_1}{}^k\nabla_{i_2}F_k{}^j + F_{i_2}{}^k\nabla_k F_{i_1}{}^j) \\
& + \sqrt{-1}(3\eta_j F_{i_1}{}^k F_{i_2}{}^l\nabla_k F_l{}^j + \eta_j\nabla_{i_1}F_{i_2}{}^j + 4F_{i_2}{}^j\nabla_j\eta_{i_1}) .
\end{aligned} \tag{3.36}
$$

Then

$$8d_1\eta = (G_{i_1 i_2} - G_{i_2 i_1})dx^{I_0(2)} . \tag{3.37}$$

As in the derivation of (3.16), (3.17) we can obtain

$$-2(\prod_{0,1}\delta\prod_{1,1}d_1\eta)_{i_1} = \prod_{0,1}\nabla^s[(\varepsilon_s^{k_1}\varepsilon_{i_1}^{k_2} - \varepsilon_{i_1}^{k_1}\varepsilon_s^{k_2} + F_s{}^{k_1}F_{i_1}{}^{k_2} - F_{i_1}{}^{k_1}F_s{}^{k_2})G_{i_1 i_2}] , \tag{3.38}$$

$$(3.39)\qquad -4(\prod_{1,0}\delta\prod_{2,0}d_1\eta)_{i_1} = \prod_{1,0}\nabla^s[(B_{s\ i_1}^{k_1k_2} + \sqrt{-1}C_{s\ i_1}^{k_1k_2})G_{k_1k_2}]\ ,$$

where $B_{s\ i_1}^{k_1k_2}$ and $C_{s\ i_1}^{k_1k_2}$ are defined in (3.14). After some calculations we can thus have

$$(3.40)\qquad \begin{aligned} 16(\Box_5\eta)_{i_1} &= 16[(\prod_{1,0}\delta\prod_{2,0} + \prod_{0,1}\delta\prod_{1,1})d_1\eta + \prod_{1,0}d\bar{\delta}_1\eta]_{i_1} \\ &= \eta_j[\nabla^sF_{i_1}{}^k(\nabla_sF_k{}^j - \nabla_kF_s{}^j) + F_{i_1}{}^rF_s{}^k\nabla^sF_r{}^l(\nabla_kF_l{}^j \\ &\quad - \nabla_lF_k{}^j) + 4F_{i_1}{}^k(\nabla^s\nabla_sF_k{}^j - \nabla^s\nabla_kF_s{}^j + \nabla_k\nabla^lF_l{}^j)] \\ &\quad + 8F_{i_1}{}^k\nabla_sF_k{}^j\nabla^s\eta_j - 8\nabla^s\nabla_s\eta_{i_1} + 4[\nabla_k, \nabla_{i_1}]\eta^k \\ &\quad + 4F_{i_1}{}^lF^{kj}[\nabla_l, \nabla_k]\eta_j + \sqrt{-1}\{\eta_j[4F_{i_1}{}^k\nabla^sF_s{}^l(\nabla_kF_l{}^j \\ &\quad - \nabla_lF_k{}^j) + F_{i_1}{}^r\nabla^sF_r{}^k(\nabla_kF_s{}^j - \nabla_sF_k{}^j) \\ &\quad + 3F_s{}^k\nabla^sF_{i_1}{}^l(\nabla_lF_k{}^j - \nabla_kF_l{}^j) + 4\nabla_{i_1}\nabla^kF_k{}^j \\ &\quad + 4F_{i_1}{}^kF_s{}^l\nabla^s(\nabla_kF_l{}^j - \nabla_lF_k{}^j)] + 8\nabla^sF_s{}^k\nabla_k\eta_{i_1} \\ &\quad + 4F_{i_1}{}^rF^{sk}(\nabla_rF_s{}^l + \nabla_sF_r{}^l)\nabla_l\eta_k - 4(\nabla_{i_1}F_j{}^k + \nabla_jF_{i_1}{}^k)\nabla_k\eta^j \\ &\quad + 4F^{kj}[\nabla_{i_1}, \nabla_k]\eta_j + 4F_{i_1}{}^l[\nabla_l, \nabla_k]\eta^k + 4F^{sk}[\nabla_s, \nabla_k]\eta_{i_1}\}\ . \end{aligned}$$

4. Realization of $\Box$'s

Theorem 4.1. *The complex Laplacian $\Box_i$, $i = 1, \cdots, 5$, for an almost-Hermitian structure is real with respect to every 0-form if and only if the structure is almost-semi-Kählerian. Moreover, with respect to every 0-form, if $\Box_i$, $i = 1, \cdots, 5$, for an almost-Hermitian structure is real, then $\Box_i = \Delta/2$ for $i = 1, \cdots, 5$.*

Proof. The theorem follows immediately from (3.1), (3.5), (3.22), (3.28), (3.33) and (2.8) by choosing the 0-form ξ to be x^k for an arbitary k with respect to any local coordinates $x^1, \cdots, x^n$.

Theorem 4.2. *For an almost-Hermitian structure, if the Laplacian $\Box_i$, $i = 1, 2$ or 4, is real with respect to all 0- and 1-forms, then the structure is Kählerian.*

Kodaira and Spencer [3] have shown that if the relation

$$(4.1)\qquad \Box_1 = \Delta/2$$

holds for an almost-Hermitian structure, then the structure is integrable. The particular case of Theorem 4.2 in which

$$(4.2)\qquad \Box_i = \Delta/2 \qquad (i = 1, 2 \text{ or } 4)$$

holds was a conjecture for some time; it was proved by Hsiung [2] for $i = 2$ and by A. W. Adler [1] for $i = 1$ by a different method under a stronger assumption that (4.1) holds for a Hermitian structure and all 0-, 1- and 2-forms. Theorem 4.2 was proved by Hsiung [2] and Ogawa [5] for $i = 2$, and by Ogawa [5] for $i = 1$ by a somewhat different method.

Proof. (i) $i = 2$. In [2, p. 148] Hsiung proved that under the assumption of the theorem the structure F satisfies[2] (2.17) and (2.28). Then the theorem follows immediately from Lemmas 2.4 and 2.3; this was pointed out to one of the authors by H. Wakakuwa.

(ii) $i = 1$. Using the Ricci and Bianchi identities and (2.23) we can easily obtain

$$F_{i_1}{}^k[\nabla_j, \nabla_k]\eta^j = F_{i_1}{}^k R_k{}^j \eta_j \,, \tag{4.3}$$

$$F^{kj}[\nabla_k, \nabla_{i_1}]\eta_j - F^{jk}[\nabla_j, \nabla_k]\eta_{i_1} = -\tfrac{1}{2}F^{kl}R^j{}_{i_1kl}\eta_j \,, \tag{4.4}$$

$$\nabla_s\nabla_{i_1}F^{sj} - \nabla_{i_1}\nabla_s F^{sj} = F_a{}^j R_{i_1}{}^a - \tfrac{1}{2}F^{sa}R^j{}_{i_1sa} \,. \tag{4.5}$$

By assumption, for any 1-form η, Im $\square_1\eta = 0$ which is reduced to, in consequence of Theorem 4.1, (2.8), (3.21), (4.3), (4.4), (4.5),

$$2(\nabla_s F_{i_1}{}^j + \nabla_{i_1}F_s{}^j)\nabla_j\eta^s + (\nabla^s\nabla_s F_{i_1}{}^j + F_{i_1}{}^k R_k{}^j - R_{i_1}{}^k F_k{}^j)\eta_j = 0 \,. \tag{4.6}$$

By choosing

$$\eta = dx^h \,, \qquad \text{for an arbitary } h \tag{4.7}$$

with respect to any local coordinates $x^1, \cdots, x^n$, from (4.6) it thus follows that

$$\nabla^s\nabla_s F_{i_1}{}^h + F_{i_1}{}^k R_k{}^h - R_{i_1}{}^k F_k{}^h = 0 \,. \tag{4.8}$$

Multiplying (4.8) by $F_h{}^{i_1}$ and using (2.1) we obtain (2.16), and therefore the structure F is Kählerian by Lemma 2.2.

(iii) $i = 4$. At a general point P of the manifold M^n we choose orthogonal geodesic local coordinates $x^1, \cdots, x^n$ so that

$$g_{ij}(P) = \delta_{ij} \,, \qquad \Gamma^k_{ij}(P) = 0 \,, \tag{4.9}$$

where δ_{ij} are Kronecker deltas. By using Theorem 4.1, and choosing η to satisfy (4.7) first and then

$$\eta = x^h dx^l \,, \qquad \text{for any fixed distinct } h \text{ and } l \tag{4.10}$$

with respect to the geodesic local coordinates $x^1, \cdots, x^n$, from (3.32) the condition Im $(\square_4\eta) = 0$ for any 1-form η is reduced to

$$\nabla^j\nabla_j F_{i_1}{}^h - \nabla^j\nabla_{i_1}F_j{}^h = 0 \,, \tag{4.11}$$

[2] By mistake, (2.28) was printed as $F_i{}^h R^i{}_{jkl} = F_j{}^i R^h{}_{ikl}$ in [2, p. 148].0

$$\nabla^h F_{i_1 l} + 2\nabla_{i_1} F_l{}^h + \nabla_l F_{i_1}{}^h = 0 . \tag{4.12}$$

Interchanging l, i_1 in (4.12) and adding the resulting equation to (4.12) we obtain

$$\nabla_{i_1} F_l{}^h + \nabla_l F_{i_1}{}^h = 0 . \tag{4.13}$$

From (4.11), (4.13) it thus follows that

$$\nabla^j \nabla_j F_{i_1}{}^h = 0 , \tag{4.14}$$

and hence by Lemma 2.2 the structure is Kählerian.

5. Relationships among $\square$'s

Theorem 5.1. *If for an almost-Hermitian structure the relation*

$$\operatorname{Im} \square_1 = \operatorname{Im} \square_i \qquad (i = 2 \text{ or } 4) \tag{5.1}$$

holds for all 0- *and* 1-*forms, then the structure is Kählerian.*

Proof. (i) $i = 2$. From (3.5), (3.1) and condition (5.1) for any 0-form ξ, we have

$$\nabla^h F_h{}^j \nabla_j \xi = 0 . \tag{5.2}$$

By choosing $\xi = x^i$ for an arbitary i with respect to any local coordinates $x^1, \cdots, x^n$, from (5.2) follows immediately (2.8), which together with (3.2), (3.21), (3.20) reduces condition (5.1) for any 1-form η to

$$(\nabla_s F_{i_1}{}^j + \nabla_{i_1} F_s{}^j) \nabla_j \eta^s - (\nabla^s \nabla_{i_1} F_s{}^j - \nabla^s \nabla_s F_{i_1}{}^j) \eta_j = 0 . \tag{5.3}$$

Choosing η to satisfy (4.7) first and then (4.10) with respect to the local coordinates $x^1, \cdots, x^n$ defined by (4.9) we therefore obtain (4.11), (4.13), and hence the structure is Kählerian for the same reasoning given in the proof (iii) of Theorem 4.2.

(ii) $i = 4$. As in part (i), from (3.5), (3.28), (3.1) and condition (5.1) for any 0-form ξ, we obtain (2.8), which together with (3.21), (3.32) reduces condition (5.1) for any 1-form η to

$$(\nabla_j F_{i_1}{}^k - \nabla^k F_{i_1 j}) \nabla_k \eta^j = 0 . \tag{5.4}$$

By choosing η to satisfy (4.10) with respect to the local coordinates $x^1, \cdots, x^n$ defined by (4.9), we have

$$\nabla_l F_{h i_1} - \nabla_h F_{l i_1} = 0 . \tag{5.5}$$

Thus by Lemma 2.1 the structure is Kählerian.

Theorem 5.2. *If for an almost-Hermitian structure either the relation*

$$\text{Im } \square_2 = \text{Im } \square_4 \tag{5.6}$$

or

$$\text{Re } \square_2 = \text{Re } \square_4 \tag{5.7}$$

holds for all 1*-forms, where* Re *denotes the real part, then the structure is Kählerian.*

Proof. From (3.1), (3.32), by the same argument as in the proof of Theorem 5.1 for $i = 4$ it is easily seen that conditions (5.6), (5.7) imply

$$\nabla_h F_{i_1}{}^l = \nabla_{i_1} F_h{}^l = 0 \,, \tag{5.8}$$

$$F_h{}^j \nabla_{i_1} F_j{}^l - F_h{}^j \nabla_j F_{i_1}{}^l = 0 \,, \tag{5.9}$$

respectively. By multiplying (5.9) by $F_k{}^h$, we can reduce (5.9) to (5.8). Hence by Lemma 2.1, the structure is Kählerian under either (5.6) or (5.7).

Theorem 5.3. *If for an almost-Hermitian structure the relation*

$$\text{Im } \square_2 = \text{Im } \square_5 \tag{5.10}$$

holds for all 0*- and* 1*-forms, then the structure is Kählerian.*

Proof. From (3.33), (3.5), (3.1) and condition (5.10) for any 0-form ξ we obtain (2.8). Then by the same argument as in the proof of Therem 5.1 for $i = 2$, (2.8), (3.2), (3.40) reduce condition (5.10) for any 1-form η to

$$\begin{aligned} &F_{i_1}{}^r \nabla^s F_r{}^k (\nabla_k F_s{}^h - \nabla_s F_k{}^h) + 3F_s{}^k \nabla^s F_{i_1}{}^l (\nabla_l F_k{}^h - \nabla_k F_l{}^h) \\ &\quad + 4F_{i_1}{}^k F_s{}^l \nabla^s (\nabla_k F_l{}^h - \nabla_l F_k{}^h) = 0 \,, \end{aligned} \tag{5.11}$$

$$F_{i_1}{}^r F^{sl} (\nabla_r F_s{}^h + \nabla_s F_r{}^h) = 0 \,. \tag{5.12}$$

Multiplying (5.12) by $F_j{}^{i_1} F_l{}^k$ and use of (2.1) give

$$\nabla_j F_k{}^h + \nabla_k F_j{}^h = 0 \,. \tag{5.13}$$

Substituting (5.13) in (5.11) we can easily obtain

$$2F_{i_1}{}^k F_s{}^l \nabla^s \nabla_k F_l{}^h - F_s{}^k \nabla^s F_{i_1}{}^l \nabla_k F_l{}^h = 0 \,. \tag{5.14}$$

Multiplying (5.14) by $F_h{}^{i_1}$ and using (2.1), (2.8), (5.13) we therefore have

$$\nabla_s F_{i_1 l} \nabla^s F^{i_1 l} = 0 \,, \tag{5.15}$$

which implies that $\nabla_s F_{i_1 l} = 0$. Hence the structure is Kählerian. q.e.d.

Finally, it should be remarked that there are no theorems involving the Laplacian $\square_3$ similar to Theorems 4.2, 5.1, 5.2, 5.3. However, we have the following two theorems, the proofs of which are omitted.

Theorem 5.4. *If for an almost-Hermitian structure the relation*

$$\text{Im}\,\square_3 = \text{Im}\,\square_1 + \tfrac{1}{2}\,\text{Im}\,(\bar{\vartheta}_1\partial_2) \tag{5.16}$$

holds for all 1-*forms, then the structure is Kählerian.*

Theorem 5.5. *If for an almost-semi-Kählerian structure the relation*

$$\text{Im}\,\square_3 = \text{Im}\,\square_i + \tfrac{1}{2}\,\text{Im}\,(\bar{\vartheta}_1\partial_2) \qquad (i = 2 \text{ or } 4) \tag{5.17}$$

holds for all 1-*forms, then the structure is Kählerian.*

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